# Computational Economics

## Lecture 7: Krussel-Smith Model

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# Introduction

In the previous lecture, we studied a model that allowed us to study the consumption-saving behavior of heterogeneous households in terms of wealth and labor income.

We are interested in building a model that combines this framework with aggregate fluctuations such that we can study the nexus between household heterogeneity and macro shocks on aggregate variables and cross-sectional outcomes.

Note: A key byproduct is that we can assess how close the predictions of the representative agent model are to those of the heterogeneous agent model.

# Overview

We will focus on the model presented in Krusell-Smith (1998) which is the first paper to combine heterogenous agent and aggregate shocks.

The economic environment is similar to that of the previous lecture, except that aggregate technology follows a stochastic process.

This modification introduces a key challenge: how to solve for the dynamics when the wealth distribution is endogenous and varies with aggregate shocks.

# The Economy

# Households

A unit mass of ex-ante identical households subject to idiosyncratic income shocks facing a common borrowing constraint.

Each household $i \in [0, 1]$ solves the following problem:

$$\max_{\{c_{it}\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^t u(c_{it}) \tag{1}$$

subject to

$$a_{it+1} + c_{it} = w_t z_{it} + (1 + r_t) a_{it} \text{ , and } a_{it} \geq B. \tag{2}$$

where $c_{it}$ is consumption, $a_{it}$ is assets, $z_{it}$ is an idiosyncratic income shock with transition matrix $Q(z_t, z_{t+1}; \Theta_t)$, $w_t$ is the wage rate, $r_t$ is the net interest rate, and $B \leq 0$ is the borrowing limit.

Note: Whenever $B < 0$, we assume that a financial intermediary collects savings and uses them to finance loans and firms' capital.

# Households (cont'd)

We solve the household problem using dynamic programming:

$$V(a, z, \Theta, \Psi) = \max_{\{c, a'\}} \left\{ u(c) + \beta E \left[ V(a', z', \Theta', \Psi') \right] \right\}$$

subject to

$$a' + c = w(\Theta, \Psi) z + (1 + r(\Theta, \Psi)) a, \qquad a' \geq B.$$

The associated policy function for savings is denoted by

$$a' = g(a, z, \Theta, \Psi).$$

# Firms

The firm side is summarized by a profit-maximizing representative firm that produces using capital and labor as inputs.

In every period, the representative firm solves

$$\max_{\{K_t, L_t\}} \Theta_t K_t^{\alpha} L_t^{1-\alpha} - w_t L_t - (r_t + \delta) K_t \tag{3}$$

where $K_t$ is aggregate capital, $L_t$ is aggregate labor, $\Theta_t$ is aggregate productivity, and $\delta$ is the capital depreciation rate.

Note: $(r_t + \delta)$ follows from non-arbitrage between investing in an asset that returns $(1 + r_t)$ or $(R_t + 1 - \delta)$.

# Firms (cont'd)

The representative firm's FOCs are given by:

$$\alpha\Theta_t K_t^{\alpha-1} L_t^{1-\alpha} = (r_t + \delta) \tag{4}$$

$$(1-\alpha)\Theta_t K_t^{\alpha} L_t^{\alpha} = w_t. \tag{5}$$

From Equation (4), we can express $r_t$ as a function of $K_t$:

$$r_t = \alpha\Theta_t \left(\frac{L_t}{K_t}\right)^{1-\alpha} - \delta. \tag{6}$$

Combining Equation (5) and Equation (6) yields an expression for $w_t$ as a function of $r_t$:

$$w_t(r_t) = \Theta_t(1-\alpha)(\alpha\Theta_t/(r_t+\delta))^{\alpha/(1-\alpha)}. \tag{7}$$

# Aggregate State of the Economy

The aggregate state of the economy at time $t$ is described by the pair $\{\Theta_t, \Psi_t\}$ where

- $\Theta_t$ is aggregate productivity that follows a Markov process.
- $\Psi_t$ is the distribution of households over asset holdings $a_{it}$ and idiosyncratic productivity $z_{it}$.

Note: Under this formulation, an agent forecasts the entire wealth distribution (an infinite-dimensional object) in order to form expectations about prices.

# Equilibrium

A Recursive Rational Expectations Equilibrium (RREE) is:

- A value function $V(a, z, \Theta, \Psi)$ and its associated policy function $g(a, z, \Theta, \Psi)$.
- A distribution $\Psi$ of households over assets and idiosyncratic productivity, and its law of motion $\Psi'(\Theta, \Psi)$.
- Functions for prices $r(\Theta, \Psi)$ and $w(\Theta, \Psi)$.

such that

1. $V(a, z, \Theta, \Psi)$ and $g(a, z, \Theta, \Psi)$ solve the households' problem taking prices as given.
2. The representative firm maximizes taking prices as given.
3. The distribution over assets and idiosyncratic productivity evolves according to $\Psi(a', z') = \int g(a, z, \Theta, \Psi)\, Q(z, z', \Theta)\, d\Psi(a, z)$.
4. Markets clear: $K_t = \int_0^1 a_{it}\, di$ and $L_t = \int_0^1 z_{it}\, di$.

# Solving the Model

# Solving the Model

The key difficulty is dealing with the presence of the distribution of households $\Psi$ as a state variable.

KS propose to solve for an approximate equilibrium in which agents only observe the aggregate capital stock $K$ rather than the full distribution $\Psi$, where $K(\Theta, \Psi)$.

They guess that the law of motion for the capital stock can be well approximated as:

$$\log(K') = b_{0j} + b_{1j} \log(K) \ , \ \forall j \in \{1, ..., n_\Theta\}$$

where $n_\Theta$ is the number of states for aggregate technology.

Note: Under this formulation, one only needs to know the mean of aggregate savings to form expectations about next period's capital, but one could include more moments in the law of motion.

# Solving the Model (cont'd)

Given the KS assumption about the law of motion for capital, one can rewrite the model in terms of $K$ instead of $\Psi$. Solving for the RREE can then be done by iterating over the following steps:

0. Guess a vector of coefficients $\mathbf{b}$ for the law of motion for $K$.
1. Solve for the policy function $g(a, z, \Theta, K)$ taking price functions as given.
2. Simulate the economy with many agents, and estimate the coefficients $\tilde{\mathbf{b}} = \{b_{0j}, b_{1j}\}_{j \in \{1,\ldots,n_\Theta\}}$ with OLS.
3. Compare $\mathbf{b}$ and $\tilde{\mathbf{b}}$. If they differ by more than a prespecified criterion, update the guess for $\mathbf{b}$ and return to step 1.

# Solving the Model (cont'd)

In general, the approximation of the law of motion for $K$ is very accurate ($R^2 > .99$).

The appeal of the KS approximation is more than computational: in practice, no one forms expectations over an entire distribution to forecast prices.

Hence, KS connects to a broader literature in behavioral and information economics in which agents cannot observe the full state of the economy.

# Solving the Model (cont'd)

Solving this class of models (HA + aggregate shocks) has been the subject of several computational papers in the last decade.

There are now several faster methods available. In general, they involve some form of linearization around the non-stochastic steady state.

In future lectures, we will look at one of them: the Sequence-Space Jacobian method.