Computational Economics

Lecture 3: Dynamic Programming II

Etienne Briand

Concordia University

Winter 2026

Motivation

In most (interesting) economic problems, the state of the world in the next periods is not known with certainty.

In such cases, agents must form expectations about the future that account for all possible states in the next periods.

Here are some examples:

- Productivity shocks.
- Unemployment shocks.
- Financial shocks.
- Etc.

Introduction

Most results from the last lecture carry over to problems in which the return function is subject to stochastic shocks and the objective is to maximize the expected discounted value of returns.

We will see that such problems are associated with a FE of the form:

$$V(x,z) = \max_{x' \in \Gamma(x,z)} \{F(x,x',z) + \beta E[V(x',z')]\}. \tag{1}$$

A complete treatment of these problems requires tools from measure theory and stochastic processes that are beyond the scope of this class (see SLP, Ch. 7–8). That said, we will try to cover them as rigorously as possible.

Measure

Measure

The additional complexity introduced by uncertainty is the need to deal with expectations. For this reason, the state space of the variables subject to exogenous shocks must be measurable.

By measurable, we mean that the space is equipped with a collection of subsets, called a σ-algebra, which is suitable for defining probabilities or integrals (see SLP, Ch. 7).

In what follows, we assume that every state we work with is measurable in the sense described above. As a result, some statements below involve a mild abuse of language, which we adopt for notational simplicity.

Measure (cont'd)

For any measurable space, there exists a function $\mu(\cdot)$ that assigns values in $\mathcal{R}^+$, interpretable as size, to the subsets allowed by the associated σ-algebra.

A measure is non-negative, assigns zero to the null set, and is countably additive.

If the measure assigns value 1 to the whole space, then $\mu(\cdot)$ is called a probability measure.

Measure (cont'd)

We are interested in evaluating expressions of the type $E[f(\cdot, z)]$, with $z \in Z$ and Z a probability space; that is, the expected value of a real-valued function.

$f(\cdot, z) : Z \to \mathcal{R}$ is said to be measurable if, for every $y \in \mathcal{R}$

$$f^{-1}(y) = \{z \in Z : f(z) = y\}.$$

If $f(\cdot, z)$ is measurable, its expected value is given by

$$E[f(\cdot, z)] = \int_Z f(\cdot, z)\, \mu(dz). \tag{2}$$

Similarly, the conditional expectations of $f(\cdot, z) \mid z \in Z_i$ is given by

$$E[f(\cdot, z) \mid z \in Z_i] = \frac{1}{\mu(Z_i)} \int_{Z_i} f(\cdot, z)\, \mu(dz), \tag{3}$$

where $\mu(Z_i) > 0$.

Stochastic Processes

Stochastic Processes

Thus far, we have not imposed restrictions on the exogenous process that the variable z follows. We have only assumed that it is a stochastic process.

A stochastic process is defined on a probability space and is associated with a function that can generate a sample path: a sequence of realizations $z^t = \{z_1, z_2, \ldots, z_t\}$.

However, the full class of stochastic processes is too large to embed directly in dynamic programming problems.

Stochastic Processes

We will restrict the variable z to follow a Markov process. This is a special class of stochastic process with a recursive structure described by transition functions.

Definition: A stochastic process is Markov if, conditional on the current state, future realizations are independent of the past.

If the conditional probabilities are time-invariant, the process is said to have stationary transitions.

Stochastic Processes

A transition function $Q(z_i, z_j)$ with an admisible pair (z_i, z_j) associated with a measurable space Z has the following properties

1. $\forall\ z_i \in Z,\ Q(z_i, \cdot)$ is a probability measure.
2. $\forall\ z_j \in Z,\ Q(\cdot, z_j)$ is a measurable function.

The interpretation of $Q(z_i, z_j)$ is the probability that the next period state is z_j, given that the current shock is z_i.

$Q(z_i, z_j) := Pr\{z_{t+1} = z_j | z_t = z_i\}$.

Stochastic Dynamic Programming

Objective

As in the previous lecture, our goal is to show that the sequential representation of problems of the type presented in the introduction admits a recursive representation, such that solving the associated functional equation yields time-invariant policy functions: mappings from any state to optimal actions.

Notation

Let X and Z be measurable spaces, and $S = (X \times Z)$ the product space such that:

- X contains all the possible values for the endogenous state.
- Z contains all the possible values for the exogenous (stochastic) state.

Let the stationary transition function Q describe the evolution of the stochastic state.

Let $\Gamma : X \times Z \to X$ be the correspondence describing the constraints.

Let $F : A \to \mathcal{R}$ be the bounded period return function with $A = \{(x, x', z) \in X \times X \times Z : x' \in \Gamma(x, z)\}$

Let $\beta \in (0, 1)$ be the time discount factor.

Thus, the givens of the problem are $X, Z, Q, \Gamma, F, \beta$

Sequential Representation

Consider the problem of choosing a sequence of actions $\{x_{t+1}\}_{t=0}^{\infty}$ that maximizes the infinite sum of expected returns.
(For brevity, we skip the formal presentation of the maximization problem.)

At period $t = 0$, the decision-maker chooses a value for x_1 given the state $s_0 = (x_0, z_0)$.

He knows that decisions in periods $t \geq 1$ depend on information not available at $t = 0$.

Thus, he cannot commit to actions beyond $t = 0$, but he can make contingency plans that depend on the realizations of shocks.

Sequential Representation (cont'd)

Definition: A plan π is an initial action $\pi_0(z_0) \in X$ and a sequence of functions $\pi_t : Z^t \to X, \quad t = 1, 2, \ldots$

The interpretation is that $\pi_t(z^t)$ specifies the value of x_{t+1} to be chosen if z^t is realized.

We make the following necessary assumptions for the SP to be well-defined:

1. $\forall\ s_0$, the set of possible plans $\Pi(s_0)$ is non-empty.
2. The infinite sum of discounted expected returns can be evaluated.

Sequential Representation (cont'd)

Define the expected discounted sum for n periods by

$$v_n(\pi, s_0) = F[x_0, \pi_0, z_0] + \sum_{t=1}^{n} \int_{Z^t} \beta^t F[\pi_{t-1}(z^{t-1}), \pi_t(z^t), z_t]\mu(dz^t) \quad (4)$$

and its limit as $n \to \infty$: $v(\pi, s_0) := \lim_{n\to\infty} v_n(\pi, s_0)$.

Thus, $v(\pi, s_0)$ is the value of the infinite sum of expected discounted returns for a plan π when the initial state is s_0.

Let $V^*(s_0) = \max_{\pi\in\Pi(s_0)} v(\pi, s_0)$ define the value associated with the optimal plan π^*.

Recursive Representation

Suppose that, corresponding to the SP, there exists an associated FE of the form:

$$V(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ F(x,x',z) + \beta \int_Z V(x',z')\, Q(z,dz') \right\} \quad (5)$$

If such a value function exists, we can define its associated policy function G as:

$$G(x,z) = \{x' \in \Gamma(x,z) : V(x,z) = F(x,x',z) + \beta \int_Z V(x',z')\, Q(z,dz')\} \quad (6)$$

Recursive Representation (cont'd)

We have two theorems connecting the SP and FE representations.

Briefly summarized, they show that:

1. Any plan π generated by G attains the $V^*(\cdot)$ in the SP.
2. Any plan π^* that attains $V^*(\cdot)$ in the SP can also be generated by G.

These reflect the Principle of Optimality in a stochastic setting.

(For the proofs, see SLP, Ch. 9)

Recursive Representation (cont'd)

Define the Bellman operator on $C(S)$, the space of bounded continuous function on S, as:

$$(Tf)(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ F(x,x',z) + \beta \int_Z V(x',z')\, Q(z,dz') \right\} \quad (7)$$

Recursive Representation (cont'd)

Under standard regularity assumptions (see SLP, Ch. 9), we have

1. $T : C(S) \to C(S)$ has a unique fixed-point, with $V(\cdot, z)$ strictly concave and increasing.
2. The policy function $G(\cdot, z) : X \to X$ is continuous and single valued.
3. Off corners, $V(\cdot, z)$ is continuously differentiable in $x \in X$.
4. For $v^n = T^n v^0$, the associted sequence of policy functions $\{gn\}$ converge uniformly to the optimal policy function G.

Recursive Representation (cont'd)

The results above extend to FE that take the form:

$$V(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ F(x,x',z) + \beta \int_Z V(\phi(x,x',z'),z')\, Q(z,dz') \right\} \qquad (8)$$

That is, the next-period state x' is determined by the functional $\phi(\cdot)$, which takes the realization z' as an argument.

Stochastic Consumption-Saving Problem

Recursive Representation

Consider our standard consumption-saving problem, but assume that production is a function of capital and stochastic productivity.

This problem has the following FE:

$$V(K, Z) = \max_{K' \in \Gamma(K,Z)} \left\{ \frac{C^{1-\gamma} - 1}{1-\gamma} + \beta E\left[V(K', Z')\right] \right\} \tag{9}$$

with

$$\Gamma(K, Z) = \{K' : 0 \leq K' \leq ZK^\alpha + (1-\delta)K\} \tag{10}$$

and

$$C := ZK^\alpha + (1-\delta)K - K'. \tag{11}$$

where Z is a Markov process with an associated stationary transition function Q.

Recursive Reprensentation (cont'd)

The first-order and the enveloppe conditions are given by:

$$-C^{-\gamma} + \beta E\left\{V_1(K', Z')\right\} = 0 \qquad \text{[FOC wrt } K'\text{]}$$

$$V_1(K, Z) = C^{-\gamma}[\alpha Z K^{\alpha-1} + (1-\delta)] \qquad \text{[Envelope condition]}$$

Combining the FOC and the envelope condition, we obtain:

$$C^{-\gamma} = \beta E\left\{(C')^{-\gamma}[\alpha Z' K'^{\alpha-1} + (1-\delta)]\right\} \qquad \text{[Stochastic Euler eq.]}$$